ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2008, Vol. 2, No. 1, pp. 100–112. © Pleiades Publishing, Ltd., 2008.
Original Russian Text © V.S. Mogilatov, A.V. Zlobinskii, 2006, published in Sibirskii Zhurnal Industrial'noi Matematiki, 2006, Vol. IX, No. 1(25), pp. 91–105.

Software for the Inductive Impulse Electrical Prospecting

V. S. Mogilatov* **and A. V. Zlobinskii**

Institute of Geophysics, pr. Akad. Koptyuga 3, Novosibirsk, 630090 Russia
Received August 15, 2005

Abstract—Some basic mathematical apparatus is considered for the electric prospecting by the method of electromagnetic field formation in the Earth. The Tikhonov method of solution is analyzed in detail.

DOI: 10.1134/S1990478908010110

The field formation method uses the response of a geoelectric medium to the modification of the source activity. The most practical implementation of this principle is a complete detachment of the source and the subsequent registration of the response. The next question that may arise in choosing a certain technique is the selection of the manner in which the source will impact the medium. Even at the most superficial glance, it seems that the medium will react rather differently to the direct current from the ground and to the excitation with the help of inductive connection with the current loop. This issue is closely connected with the existence of an electromagnetic field in a stratified medium represented as the superposition of two components of different polarization (H-field and E-field, H-mode and E-mode, TE-field and TM-field, and so on). These components correspond to the transition processes of two kinds; the contribution of each of them into the resulting field depends on different properties of the extraneous current (source). Our task will be to show, in the clearest way, the connection between the configuration of the extraneous current and the type of the excited field; and, on a strictly formal basis, distinguish the part of the total inductive electromagnetic process that is excited inductively; that is, the part related to the impulse inductive sounding.

1. GENERAL SOLUTION TO THE PROBLEM OF INDUCTIVE SOUNDING

The leading theme of the suggested approach is a clear separation of the field of an arbitrary exciter into two, practically independent phases (though we restrict ourselves here to an arbitrary two-dimensional horizontal distribution of the extraneous current). Needless to say, the fact itself has been known for a long time. We can refer to the papers by V. I. Dmitriev [1], L. A. Tabarovskii [2], L. L. Van'yan [3], and J. R. Wait [4]. Clarifying the approach, we say that an ungrounded loop is a purely inductive source exciting only a TE-field (transverse electric field) in a stratified Earth; a so-called *circular electric dipole* (CED [5, 6]) is a purely galvanic source exciting a TM-field (transverse magnetic field); and, finally, a horizontal electric (grounded) dipole is a mixed source, which, in fact, is composed of three sources, i.e., a current segment (an inductive source) and the two single-point grounds (galvanic sources). These basic types of the sources are shown in Fig. 1.

Consider a one-dimensional piecewise-homogeneous geoelectric model shown in Fig. 1,a. The whole variety of different energizing devices, situated on the daylight surface ($z = z_1 = 0$) or any other boundary ($z = z_i$) and formed by the wire segments and grounds, can be formally described by introducing the distribution of the surface density (in A/m) of the extraneous current changing synchronously, i.e., $\mathbf{j}^{ex}(x, y)q(t)$. Note that the assumption about the synchronism of the current change at each point of the distributed source may be physically incorrect when we consider the ultra-early

*E-mail: VMogilat@uiggm.nsc.ru

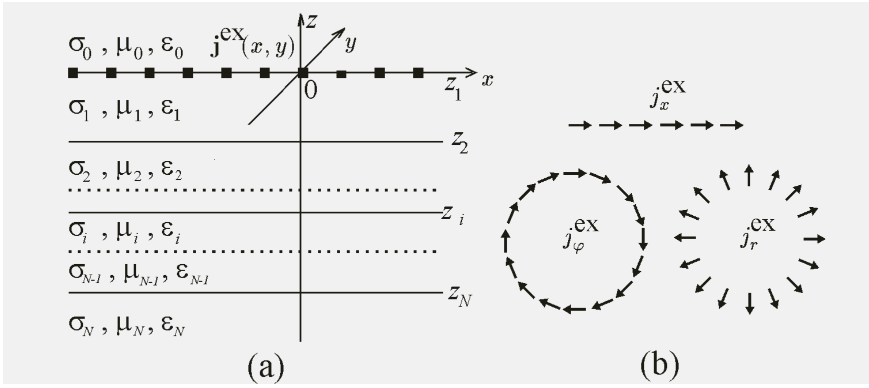

Fig. 1. A model of the medium and three types of sources

times (and it is necessary to take into account the electrical induction current and the finite propagation velocity). In each homogeneous layer ($i = 0, 1, \ldots, N$), we must solve the Maxwell system of equations:

$$\operatorname{rot} \mathbf{H} = \sigma_i \mathbb{E} + \varepsilon_i \frac{\partial \mathbf{E}}{\partial t}, \tag{1}$$

$$\operatorname{rot} \mathbf{E} = -\mu_i \frac{\partial \mathbf{H}}{\partial t}, \tag{2}$$

$$\operatorname{div} \mathbf{E} = 0, \tag{3}$$

$$\operatorname{div} \mathbf{H} = 0. \tag{4}$$

On the boundaries between the layers, the horizontal components of the field (H_x, H_y, E_x, E_y) are continuous. On the boundary containing the extraneous surface current (suppose that it is the lth boundary, for $z = z_l$), some special conditions following from (1) must hold:

$$\begin{gathered} [H_x]|_{z=z_l} = j_y^{ex}(x,y)q(t), \qquad [H_y]|_{z=z_l} = -j_x^{ex}(x,y)q(t), \\ [E_x]|_{z=z_l} = 0, \qquad [E_y]|_{z=z_l} = 0. \end{gathered} \tag{5}$$

From now on, $[F]|_{z=z_i}$ denotes the jump of F across the boundary $z = z_i$. Note that while taking (5) in this form, we disregard the influence of material carrier of the extraneous current as part of the geoelectric medium.

Let us reduce the number of unknown functions in (1)–(5) by reducing the problem to finding the vertical components of the field (this is a well-known stratagem (for example, see [2]). By (1)–(4), we can obtain the following expressions of the horizontal components in terms of the vertical components:

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \sigma_i E_z + \varepsilon_i \frac{\partial E_z}{\partial t}, \tag{6}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_i \frac{\partial H_z}{\partial t}, \tag{7}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = -\frac{\partial H_z}{\partial z}, \tag{8}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = -\frac{\partial E_z}{\partial z}. \tag{9}$$

We now formulate a problem for E_z and H_z. In each layer, they must satisfy

$$\Delta F = \mu_i \sigma_i \frac{\partial F}{\partial t} + \mu_i \varepsilon_i \frac{\partial^2 F}{\partial t^2}. \tag{10}$$

On the boundaries ($z = z_i$, $i = 1, 2, \ldots, N$; the source is on the boundary $z = z_l$), taking into account (5)–(9), we have

$$\left[\sigma E_z + \varepsilon \frac{\partial E_z}{\partial t}\right]\Big|_{z=z_i} = \begin{cases} -\operatorname{div} \mathbf{j}^{ex} q(t), & i = l, \\ 0, & i \neq l, \end{cases} \tag{11}$$

$$\left[\frac{\partial E_z}{\partial z}\right]\Big|_{z=z_i} = 0, \tag{12}$$

$$[\mu H_z]|_{z=z_i} = 0, \tag{13}$$

$$\left[\frac{\partial H_z}{\partial z}\right]\Big|_{z=z_i} = \begin{cases} -\operatorname{rot}_z \mathbf{j}^{ex} q(t), & i = l, \\ 0, & i \neq l. \end{cases} \tag{14}$$

We must add to (10)–(14) the radiation conditions for E_z and H_z.

Note now we have already obtained an important result. The problem with an arbitrary planar source was separated into two independent boundary value problems for scalar functions depending in absolutely different ways on the given distribution of the extraneous current.

We will solve these problems by separation of the variables. Since the distribution $\mathbf{j}^{ex}(x, y)$ is still arbitrary, we separate the variables on using the two-dimensional Fourier transform with respect to coordinates x and y which is defined as follows:

$$f(x, y, z) = \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f^*(\xi, \eta, z) e^{\mathbf{i}\xi x} \mathbf{e}^{\mathbf{i}\eta y}\, d\xi d\eta, \tag{15}$$

$$f^*(\xi, \eta, z) = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(x, y, z) e^{-\mathbf{i}\xi x} e^{-\mathbf{i}\eta y}\, dx dy. \tag{16}$$

In the axial symmetric case, when f depends only on $r = \sqrt{x^2 + y^2}$, the two double Fourier transforms are equivalent to the two Hankel transforms:

$$f(r, z) = \frac{1}{2\pi} \int\limits_0^{\infty} f^*(\lambda, z) J_0(\lambda r) \lambda\, d\lambda, \tag{17}$$

$$f^*(\lambda, z) = 2\pi \int\limits_0^{\infty} f(r, z) J_0(\lambda r) r\, dr, \tag{18}$$

where $\lambda = \sqrt{\xi^2 + \eta^2}$.

It is easy that by defining

$$\begin{aligned} E_z^*(z, t, \xi, \eta) &= \frac{1}{2\sigma_i} V(z, t, \lambda) D^*(\xi, \eta), \\ H_z^*(z, t, \xi, \eta) &= \frac{1}{2\lambda} X(z, t, \lambda) R^*(\xi, \eta), \end{aligned} \tag{19}$$

where

$$D^* = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \operatorname{div} \mathbf{j}^{ex}(x, y) e^{-\mathbf{i}\xi x} e^{-\mathbf{i}\eta y}\, dx dy, \tag{20}$$

$$R^* = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \operatorname{rot}_{\,z} \mathbf{j}^{ex}(x, y) e^{-\mathbf{i}\xi x} e^{-\mathbf{i}\eta y}\, dx dy, \tag{21}$$

we reduce the problem to finding the two functions X and V absolutely independent of one another and the source configuration and satisfying the same equation in each homogeneous layer ($i = 0, 1, \dots, N$)

$$\frac{\partial^2 F}{\partial z^2} - \lambda^2 F = \mu_i \sigma_i \frac{\partial F}{\partial t} + \mu_i \varepsilon_i \frac{\partial^2 F}{\partial t^2},$$

but different conditions on the boundaries ($z = z_i$, $i = 1, 2, \dots, N$):

$$[\mu X]|_{z=z_i} = 0, \qquad [X'_z]\big|_{z=z_i} = \begin{cases} -2\lambda q(t), & i = l, \\ 0, & i \neq l, \end{cases}$$

$$\left[V + \frac{\varepsilon}{\sigma}\frac{\partial V}{\partial t}\right]\bigg|_{z=z_i} = \begin{cases} -2q(t), & i = l, \\ 0, & i \neq l, \end{cases} \qquad [V'_z/\sigma]\big|_{z=z_i} = 0, \tag{22}$$

$$X \to 0, \qquad V \to 0, \qquad |z| \to \infty.$$

We will solve these boundary value problems later. Now, let us discuss a general form of the solution (the Fourier image) for all components. Applying the Fourier transform to (6)–(9) and expressing the images of the horizontal components in terms of the images of the vertical components, we infer in each layer

$$H_x^* = \frac{\bar{\eta}}{2}\left[V + \frac{\varepsilon_i}{\sigma_i}\frac{\partial V}{\partial t}\right] D^* + \frac{\bar{\xi}}{2\lambda}\frac{\partial X}{\partial z} R^*, \tag{23}$$

$$H_y^* = -\frac{\bar{\xi}}{2}\left[V + \frac{\varepsilon_i}{\sigma_i}\frac{\partial V}{\partial t}\right] D^* + \frac{\bar{\eta}}{2\lambda}\frac{\partial X}{\partial z} R^*, \tag{24}$$

$$H_z^* = \frac{1}{2\lambda} X R^*, \tag{25}$$

$$E_x^* = \frac{\bar{\xi}}{2\sigma_i}\frac{\partial V}{\partial z} D^* - \frac{\bar{\eta}\mu_i}{2\lambda}\frac{\partial X}{\partial t} R^*, \tag{26}$$

$$E_y^* = \frac{\bar{\eta}}{2\sigma_i}\frac{\partial V}{\partial z} D^* + \frac{\bar{\xi}\mu_i}{2\lambda}\frac{\partial X}{\partial t} R^*, \tag{27}$$

$$E_z^* = \frac{1}{2\sigma_i} V D^*, \tag{28}$$

where $\bar{\xi} = \mathbf{i}\xi/\lambda^2$, $\bar{\eta} = \mathbf{i}\eta/\lambda^2$, $i = 0, 1, \dots, N$. Thus, the time behavior of the field is described by the two independent functions V and X of different types. In other words, the formation process for a field from an arbitrary source is a superposition of two different processes. The contribution of each process is determined by the coefficients D^* and R^*; i.e., according to (20) and (21), by the configuration of the source (extraneous current) with the help of the values of div $\mathbf{j}^{ex}(x, y)$ and $\mathrm{rot}_z \mathbf{j}^{ex}(x, y)$. Recalling the physical meaning of divergence and rotor, it becomes clear that one component is determined by the drains or, in our case, the current draining from (flowing in) the grounds (it is excited galvanically); whereas the other component depends on the rotational component in the distribution of the extraneous current (it is excited inductively). Using the above representation of the solution for an arbitrary source, it is possible to purposefully change the source configuration so as to achieve the suppression of either galvanic or inductive component. However, the suppression of the galvanic component of the process, notably, the complete suppression (everywhere div $\mathbf{j}^{ex}(x, y) = 0$), has been known for a long time and has been used: namely, this is a nongrounded current loop.

Example 1. A current loop as an inductive source. Let us consider a special case; i.e., the distribution of the extraneous current with the azimuth symmetry. There are also many possibilities here, but we assume that, in the polar system of coordinates, there is only $j_\varphi^{ex}(r)$, and, at that, $j_\varphi^{ex}(r) = I\delta(r - a)$. This means that a circular current loop (of radius a) is under consideration. Thus, in the cylindrical coordinate system, we have

$$\operatorname{div}\mathbf{j}^{ex} = \frac{1}{r}\frac{\partial j_\varphi^{ex}}{\partial\varphi} = 0,$$

$$\operatorname{rot}_z\mathbf{j}^{ex} = \frac{1}{r}\frac{\partial(rj_\varphi^{ex})}{\partial r} = I[\delta(r-a)/r + \delta'(r-a)].$$

Hence, $D^* = 0$, and this source is purely inductive (we were suspecting this, of course).

As far as the function R^* is concerned, we will obtain the following by using the azimuth symmetry and moving from the Fourier transform (15), (16) to the Hankel transform (17), (18), and also recalling the definition of the Dirac delta-function and its derivatives:

$$R^* = 2\pi I\int_0^\infty [\delta(r-a)/r + \delta'(r-a)]J_0(\lambda r)r\,dr = 2\pi I\lambda a J_1(\lambda a). \tag{29}$$

Accounting for (23)–(28), it is easy that the components H_r, H_z, and E_φ are different from zero in the cylindrical system of coordinates. For instance, E_φ can be represented as

$$E_\varphi(r,z,t) = \frac{M_z\mu_i}{2\pi a}\int_0^\infty J_1(\lambda r)J_1(\lambda a)\,\frac{\partial X(z,t,\lambda)}{\partial t}\,d\lambda, \tag{30}$$

where $M_z = I\pi a^2$ is defined as the source moment. In theory, a loop is usually considered of an infinitely small radius but with a finite moment; i.e., a vertical magnetic dipole. In this case,

$$J_1(\lambda a) \simeq \lambda a/2,$$

and (30) will assume the form

$$E_\varphi(r,z,t) = \frac{M_z\mu_i}{4\pi}\int_0^\infty J_1(\lambda r)\lambda\frac{\partial X(z,t,\lambda)}{\partial t}\,d\lambda. \tag{31}$$

Example 2. A grounded line as a mixed source. We now consider a classical source of the transient electromagnetic field such as a grounded horizontal electric line or dipole. Take a short line with the current I grounded at the points along the x axis at $x = -dx_0/2$ and $x = dx_0/2$. Thus, the extraneous current has only the component j_x^{ex}, and, moreover,

$$j_x^{ex}(x,y) = I\delta(y)[U(x+dx_0/2) - U(x-dx_0/2)]$$

(U is the Heaviside function); whereas for the dipole with the moment Idx_0 we have

$$j_x^{ex}(x,y) = Idx_0\delta(y)\delta(x).$$

Then,

$$\operatorname{div}\mathbf{j}^{ex} = Idx_0\delta(y)\delta'(x), \tag{32}$$

$$\operatorname{rot}_z\mathbf{j}^{ex} = -Idx_0\delta'(y)\delta(x). \tag{33}$$

We further find that

$$D^* = Idx_0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(y)\delta'(x)e^{-\mathbf{i}\xi x}e^{-\mathbf{i}\eta y}\,dxdy = Idx_0\mathbf{i}\xi, \tag{34}$$

$$R^* = -Idx_0\mathbf{i}\eta. \tag{35}$$

Finally, by (23)–(28) and (15)–(18), we obtain the expressions for all components of the nonstationary field of the horizontal electric dipole:

$$
\begin{gathered}
H_x = Idx_0 \frac{\partial^2}{\partial x \partial y} \widehat{H} \left(\frac{V}{\lambda} + \frac{\varepsilon_i}{\sigma_i} \frac{V_t'}{\lambda} - \frac{X_z'}{\lambda^2} \right), \\
H_y = -Idx_0 \left[\frac{\partial^2}{\partial x^2} \widehat{H} \left(\frac{V}{\lambda} + \frac{\varepsilon_i}{\sigma_i} \frac{V_t'}{\lambda} \right) + \frac{\partial^2}{\partial y^2} \widehat{H} \left(\frac{X_z'}{\lambda^2} \right) \right], \qquad H_z = Idx_0 \frac{\partial}{\partial y} \widehat{H}(X), \\
E_x = Idx_0 \left[\frac{\partial^2}{\partial x^2} \widehat{H} \left(\frac{V_z'}{\sigma_i \lambda} \right) + \frac{\partial^2}{\partial y^2} \widehat{H} \left(\frac{\mu_i X_t'}{\lambda^2} \right) \right], \\
E_y = Idx_0 \frac{\partial^2}{\partial x \partial y} \widehat{H} \left(\frac{V_z'}{\sigma_i \lambda} - \frac{\mu_i X_t'}{\lambda^2} \right), \qquad E_z = -Idx_0 \frac{\partial}{\partial x} \widehat{H} \left(\lambda \frac{V}{\sigma_i} \right),
\end{gathered}
\tag{36}
$$

where $\widehat{H}$ is the integral operator

$$
\widehat{H}(F) = \frac{1}{4\pi} \int_0^\infty J_0(\lambda r) F(\lambda) \, d\lambda,
$$

whereas X and V are the solutions of the boundary value problems (22).

Thus, in this paper we will consider in detail the solutions only for a TE-polarized field which is used in the impulse electric prospecting with the inductive excitation by a closed current loop or under the excitation, for example, by an electric grounded line but with registration of the vertical magnetic component (i.e., that part of the field which is excited inductively). In the theoretical aspect, we focus attention on the solution and the properties of the boundary value problem (22) only for the function X.

2. TWO METHODS FOR SOLVING THE FIELD FORMATION PROBLEM

We solve the boundary value problem (22) by performing some further separation of the variables. We will seek X as a superposition of the solutions of the form $Z(z)\exp(-\alpha t)$, where $\operatorname{Re} \alpha \geq 0$. Define the function Z as $Z(z) = A\zeta(z)$ above the boundary with the extraneous current ($z \geq z_l$) and as $Z(z) = B\zeta(z)$ below this boundary. The function ζ can be expressed in the ith layer in terms of its own (interior) values on the lower or upper boundary of the layer. Putting $\zeta_i = \zeta(z_i)$ and $\zeta_i' = \zeta_z'(z_i)$ for $i = 1, 2, \ldots, N$, we have in each layer ($z_i \geq z \geq z_{i+1}$)

$$
\begin{aligned}
\zeta(z) &= \zeta_1 \exp(-u_0 z), \qquad z \geq 0 \quad \text{(in the air)}, \\
\zeta(z) &= \zeta_i \operatorname{ch}[u_i(z - z_i)] + \frac{\zeta_i'}{u_i} \operatorname{sh}[u_i(z - z_i)]
\end{aligned}
\tag{37}
$$

or

$$
\begin{gathered}
\zeta(z) = \zeta_{i+1} \operatorname{ch}[u_i(z - z_{i+1})] + \frac{\zeta_{i+1}'}{u_i} \operatorname{sh}[u_i(z - z_{i+1})], \\
\zeta(z) = \zeta_N \exp[u_N(z - z_N)], \qquad z \leq z_N.
\end{gathered}
$$

Here $u_i^2 = \lambda^2 + k_i^2$, where $k_i^2 = -\alpha\mu_i\sigma_i + \alpha^2\mu_i\varepsilon_i$ and $i = 0, 1, \ldots, N$. The functions

$$
f = \mu\zeta, \qquad h = \zeta_z'/\lambda
$$

are continuous across the simple boundary.

On this stage, the form of the parameter α needs to be specified. Usually, it is taken as $\mathbf{i}\omega$, and X is represented by the Fourier integral

$$
X = \widehat{F}(Z) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(z) e^{-\mathbf{i}\omega t} \, d\omega. \tag{38}
$$

We find the function Z having the source on the lth boundary and determining the coefficients A and B, while taking into account, by (22), the conditions on this boundary, which are now assigned to Z:

$$Z(z) = \frac{2Q(\omega)\hat{f}_l}{\check{f}_l\hat{h}_l - \hat{f}_l\check{h}_l}\,\check{\zeta}(z), \quad z \geq z_l \quad \text{(over the source)}, \tag{39}$$

$$Z(z) = \frac{2Q(\omega)\check{f}_l}{\check{f}_l\hat{h}_l - \hat{f}_l\check{h}_l}\,\hat{\zeta}(z), \quad z \leq z_l \quad \text{(under the source)}. \tag{40}$$

Here, the notation $\check{x}$ means that the function x is determined above the source successively from top to bottom by (37), and, at that, the value on the upper side of the first boundary is arbitrary. Correspondingly, $\hat{x}$ signifies that x is determined from below from the lower side of the last boundary. The index indicates that the value is taken on the corresponding boundary. The function $Q(\omega)$ is a transformant of the function that describes the form of the exciting impulse; i.e, $q(t) = \widehat{F}(Q)$. For the instantaneous turning on, we have, in particular, $Q(\omega) = 1/(-\mathbf{i}\omega)$. If $q(t) = \exp(-\mathbf{i}\omega_0 t)$ (i.e., $Q(\omega) = \delta(\omega - \omega_0)$) then we obtain a harmonic solution for the frequency ω_0.

In fact, performing the separation of variables, we used, in a somewhat unusual order, the conventional method of solving the field formation problem known as the frequency domain method. It was proposed most soundly in [7]. The problems of numerical implementation of this algorithm have been solved, but only in a quasistationary approximation. Considering the displacement current sharply diminishes the possibility of numerical realization of the Fourier transform.

In the quasistationary approximation, we can suggest another method for solving the boundary value problems (22), which was used for the first time by A. N. Tikhonov in [8]. If we disregard the displacement currents and take the upper (air) and lower half-spaces to be insulating then the solution for X and V in those regions has the form $F(z) = C\exp(-\lambda|z|)$; and the problems for X and V can be restricted with respect to z ($0 \geq z \geq z_N$) by replacing the boundary conditions at $z = 0$ and $z = z_N$ with the closing conditions. First, we solve the problem for a turning-off regime (for $t = 0$) by reducing it to a problem with an initial condition. Instead of (22), we thus obtain the next problem for X which is the only one of interest for us here:

$$\begin{gathered} X''_{zz} - \lambda^2 X = \mu\sigma\frac{\partial X}{\partial t}, \qquad z_i > z > z_{i+1}, \quad i = 1, \ldots, N-1, \\ X'_z + \lambda\frac{\mu_1}{\mu_0}X = 0, \qquad z = 0, \\ [\mu X] = 0, \quad \left[X'_z\right] = 0, \qquad z = z_2, \ldots, z_{N-1}, \\ X'_z - \lambda\frac{\mu_{N-1}}{\mu_N}X = 0, \qquad z = z_N, \\ X = \overline{X}(\lambda, z), \qquad t = 0; \qquad X = 0, \quad t = \infty, \end{gathered} \tag{41}$$

where $\overline{X}(\lambda, z)$ is the solution for a constant current. Note that the initial condition for X is the distribution (Fourier-image) of the magnetic field of a constant extraneous current.

In this case, the parameter α can assume a discrete set of real values $\alpha_j \geq 0$, and the solution X is represented as the Fourier series

$$X = \sum_{j=0}^{\infty} C_j\zeta_j(z)T_j(t), \tag{42}$$

where $T_j(t) = \exp(-\alpha_j t)$ for the instantaneous switching on. However, if the current in the source changes as $q(t)$ then

$$T_j(t) = -\int\limits_{-\infty}^{t} q(\tau)\alpha_j\exp\left[-\alpha_j(t - \tau)\right]d\tau.$$

We determine $\zeta_j(z)$ successively from top to bottom by setting $\zeta'_{j1} = \lambda$, taking (37) into account, and satisfying the boundary conditions in (41). The condition on the lower boundary serves as an equation to determine α_j. In the case of a superconducting base, the condition on the lower boundary for X is $\zeta_{jN} = 0$. The coefficients C_j are determined from the initial conditions on X and the orthogonality of the functions $\mu\sqrt{\sigma}\zeta_j$ in the domain $0 \geq z \geq z_N$. It should be also noted that $\overline{X}(\lambda, z)$ satisfies the equation

$$F''_{zz} - \lambda^2 F = 0$$

and the same conditions on the boundaries as X. Thus,

$$C_j = \frac{2\lambda f_{j1}}{\alpha_j \sum\limits_{i=1}^{N-1} M_{ji}\mu_i^2\sigma_i}, \tag{43}$$

where in each layer we determine

$$M_{ji} = \int\limits_{z_i}^{z_{i+1}} [\zeta_j(z)]^2\, dz = \frac{1}{2u_{ji}^2}\big[d_i\big(\zeta_{ji}^2\, u_{ji}^2 - \big[\zeta'_{ji}\big]^2\big) + \big(\zeta_{ji+1}\zeta'_{ji+1} - \zeta_{ji}\zeta'_{ji}\big)\big],$$

and the continuous functions f are defined in (36), whereas $d_i = z_i - z_{i+1}$ is the thickness of the ith layer ($i = 1, 2, \dots, N-1$).

Note that $u_{ji} = \sqrt{\lambda^2 - \alpha_j\mu_i\sigma_i}$ may assume imaginary values, in which case a real hyperbolic solution (37) changes into a real trigonometric solution. The real arithmetic of this algorithm allows us to implement some fairly rapid numerical tasks.

Thus, we have solved the boundary value problem for X. Here we have described, in a rather concise form, a one-dimensional mathematical apparatus, with the help of which we can construct an algorithm for calculating a transient field of an inductive source on the daylight surface (or on any other boundary); and, furthermore, the calculation can be performed by two methods whose joint usage allows us to organize some universal, reliable, and fast computational procedures (as it is implemented in the PODBOR complex).

3. THE TIKHONOV SOLUTION

A solution of the stabilization problem by the two methods was briefly described above. This solution is nonstandard in many details and even more so in their combination. Recall that the source was taken into account as a boundary condition, the number of functions to determine was diminished without introducing the potentials, and the solution was started with transition to a nonstationary problem in a domain of spatial harmonics (the classical approach presupposes the introduction of some vector-potentials for the point sources and transition to a frequency regime). Further, a one-dimensional nonstationary problem was solved in a domain of spatial harmonics by the two methods of separation of variables, and the Fourier integral as well as the Fourier series were obtained (in the quasistationary approximation and under infinite or zero resistance of the base). However, abstracting from the details and the order of application of the transformations, we should admit that the solutions obtained here (though more convenient, compact, and general) correspond nevertheless to the two classical solutions known as the solution in a frequency domain and the solution in a time domain. As it is known, these two main approaches to the problem of field formation in a stratified medium were suggested almost simultaneously by A. N. Tikhonov [8, 9] and S. M. Sheinman [7].

The Sheinman method leads to a double Fourier–Hankel integral. The algorithm used by Sheinman has become the most widespread and developed both in Russia and abroad (for example, see [3, 4]) despite the difficulties in its numerical realization which are connected with the oscillating factors in the Fourier and Hankel transforms. Indeed, this method is clear and natural; and the solution is universal and includes, as an intermediate stage, the calculation of a usual frequency regime which is, in its own right, a working regime of some electromagnetic methods. So, the difficulties of numerical realization are overcome somehow, but only in the quasistationary approximation. The most significant numerical realization of this approach in Russia is a code by L. A. Tabarovskii and V. P. Sokolov [10], where the spline interpolation was used in integration; and, abroad, there are the codes by W. L. Anderson, in which he used his digital filters method [11–14].

However, the solution of the field formation problem in the form of a double Fourier–Hankel integral has an additional shortcoming; namely, the asymptotic developments are difficult to obtain. Based on this, it is fairly difficult to conduct the analysis of the singularities of formation of the fields from various sources, especially, in the media with insulating bases.

The Tikhonov method was developed for some time by O. A. Skugarevskaya, P. P. Frolov, and V. I. Dmitriev [15, 16], but this did not result in the creation of an efficient computational procedure for a multilayer medium. This solution is almost completely unknown abroad (we can only refer to [17]) and has been rarely applied in this country. It should be noted that the Tikhonov solution that we discuss here is, by no means, the unique representation of the solution of the field formation problem in a time domain. A. N. Tikhonov himself proposed another solution in [18] by the multiple reflections method which is convenient at the early stage. A general approach in a time domain is to move to the one-dimensional nonstationary or quasistationary problem in a domain of spatial harmonics. Therefore, this approach is sometimes called the *method of stabilizing spatial harmonics*. This one-dimensional problem can be solved by various means. In [19] it is solved by a finite-difference method. The authors of [20] separated the variables t and z in the boundary value problem and reduced the latter to a Sturm problem which they suggested to solve numerically as well. In particular, the Tikhonov approach itself consists in solving a Sturm problem successively and analytically, obtaining and analyzing the equation for the eigenfunctions, representing the solution as a series in eigenfunctions, and expressing the coefficients of this representation analytically basing on the initial condition.

There is another possible approach to finding a solution in a time domain which is based on approximation of the geoelectric model. For the magnetic mode excited by a current loop on the daylight surface, we can describe a vertically-continuous, horizontally-stratified geoelectric medium by a discrete set of the conducting planes with sufficient precision. The solution in this case possesses some characteristics favorable for numerical calculations [21, 22].

Coming back to the original paper by A. N. Tikhonov [8], we note that a fairly specific case was presented there of a two-layer medium which was not easy to generalize. However, we were able to successfully use this method of solution as a very efficient tool for obtaining the asymptotic expressions for the late stages of stabilization of the fields of various sources [23–25]. We also succeeded in the numerical realization of a multilayer algorithm for computing the stabilization processes of electric and magnetic types under excitation by various sources [5, 6, 26]. Anyway, the Tikhonov approach into which development we invested much effort needs an up-to-date exposition.

Let us describe in detail the solution of the field formation problem by the method that was suggested by A. N. Tikhonov in [8]. In the context of the present paper, it suffices to obtain the solution for the magnetic water, which is what we will do using a current loop as the source. Consider an arbitrary horizontally-stratified model (in which $\sigma_0, \sigma_1, \dots, \sigma_{N-1}, \sigma_N$ are the conductivities, $\mu_0, \mu_1, \dots, \mu_{N-1}, \mu_N$ are the magnetic permeabilities, $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{N-1}, \varepsilon_N$ are the dielectric permeabilities, $z_1 = 0 > z_2 > \dots > z_{N-1} > z_N$ are the coordinates of the boundaries). On the boundary $l\,(z = z_l)$, we put a horizontal current loop of radius a, whose current is changing according to the law $Iq(t)$.

From (23)–(28) it is not difficult to see that, in the cylindrical system, the components H_r, H_z, and E_φ are different from zero and can be represented as

$$H_r(r,z,t) = -\frac{I}{2}\int\limits_0^\infty J_1(\lambda r)aJ_1(\lambda a)\frac{\partial X(z,t,\lambda)}{\partial z}\,d\lambda,$$

$$H_z(r,z,t) = \frac{I}{2}\int\limits_0^\infty J_0(\lambda r)aJ_1(\lambda a)\lambda X(z,t,\lambda)\,d\lambda, \tag{44}$$

$$E_\varphi(r,z,t) = \frac{I\mu(z)}{2}\int\limits_0^\infty J_1(\lambda r)aJ_1(\lambda a)\frac{\partial X(z,t,\lambda)}{\partial t}\,d\lambda.$$

In the general case, we have the following boundary value problem for X:

$$\begin{gathered}\frac{\partial^2 X}{\partial z^2} - \lambda^2 X = \mu\sigma\frac{\partial X}{\partial t} + \mu\varepsilon\frac{\partial^2 X}{\partial t^2}, \qquad z_i > z > z_{i+1}, \quad i = 1, \dots, N-1, \\ [\mu X] = 0, \qquad [X'_z] = -2\lambda q(t)\Delta(i-l), \qquad i = 1, \dots, N, \\ X \to 0, \quad |z| \to \infty,\end{gathered} \tag{45}$$

where $\Delta(i-l) = 1$ for $i = l$ and $\Delta(i-l) = 0$ for $i \neq l$.

As we have found out above, this problem can be solved by separation of variables and singling out the variable t in the form $\exp(-\mathbf{i}\omega t)$, where the continuous variable ω assumes a continuous spectrum of values $-\infty \leq \omega \leq \infty$, which leads us to a traditional solution of the stabilization problem known as the solution in a frequency domain.

Right now, however, we are interested in the solution proposed by A. N. Tikhonov. To avoid details, we consider the medium to be nonmagnetic ($\mu = \mu_0$ everywhere, where μ_0 is the vacuum magnetic permeability).

The first step consists in restricting the problem with respect to z to the domain $0 \geq z \geq z_N$. To this end, adopt the quasistationary approximation ($\varepsilon = 0$ everywhere) and assume the resistances of the upper and lower half-spaces to be equal to infinity ($\sigma_0 = \infty$ and $\sigma_N = \infty$). In these regions, the right-hand side of the equation for X turns to zero, and the solution in the upper half-space is

$$X(z,t) = X(0,t)\exp(-\lambda z).$$

In the lower half-space, the solution is correspondingly

$$X(z,t) = X(z_N,t)\exp[\lambda(z - z_N)].$$

Approaching the boundaries, we obtain at the limit on the boundaries

$$X'_z(0,t) + \lambda X(0,t) = 0, \qquad X'_z(z_N,t) - \lambda X(z_N,t) = 0.$$

However, these are the conditions on the outer sides of the boundaries of the domain $0 \geq z \geq z_N$. Taking into account the boundary conditions for X in (45), we infer the conditions on the inner sides of the boundaries of the domain:

$$X'_z(0,t) + \lambda X(0,t) = -2\lambda q(t)\Delta(1-l), \qquad X'_z(z_N,t) - \lambda X(z_N,t) = -2\lambda q(t)\Delta(N-l).$$

Further, let us simplify the problem by considering a source regime such as switching off a constant current at the moment $t = 0$ (i.e., $q(t) = 1 - U(t)$, where $U(t)$ is the Heaviside function). This allows us to consider the stationary problem separately and use its solution as an initial condition for the quasistationary case. Thus, we have the stationary problem with a source in the bounded domain $0 \geq z \geq z_N$:

$$\begin{gathered}\overline{X}''_{zz} - \lambda^2\overline{X} = 0, \qquad z_i > z > z_{i+1}, \qquad i = 1, \dots, N-1, \\ \overline{X}'_z + \lambda\overline{X} = -2\lambda\Delta(1-l), \qquad z = z_1 = 0, \\ [\overline{X}] = 0, \qquad [\overline{X}'_z] = -2\lambda\Delta(i-l), \quad z = z_i, \\ \overline{X}'_z - \lambda\overline{X} = -2\lambda\Delta(N-l), \qquad z = z_N,\end{gathered} \tag{46}$$

and the quasistationary problem in the time domain $t \geq 0$, but without a source:

$$\begin{gathered}X''_{zz} - \lambda^2 X = \mu_0\sigma\frac{\partial X}{\partial t}, \qquad z_i > z > z_{i+1}, \qquad i = 1, \dots, N-1, \\ X'_z + \lambda X = 0, \qquad z = 0, \\ [X] = 0, \qquad [X'_z] = 0, \qquad z = z_i, \\ X'_z - \lambda X = 0, \qquad z = z_N, \\ X = \overline{X}(\lambda, z), \qquad t = 0; \qquad X = 0, \qquad t = \infty.\end{gathered} \tag{47}$$

Separate the variables in the boundary value problem (47) in a rather obvious way as $\zeta(z)\exp(-\alpha t)$ requiring $\alpha > 0$. Then, for $\zeta(z)$, we arrive at a boundary value problem of the third kind of a Sturm type:

$$\zeta''_{zz} - (\lambda^2 - \mu_0\sigma\alpha)\zeta = 0, \qquad z_i > z > z_{i+1}, \qquad i = 1, \dots, N-1, \tag{48}$$

$$\zeta'_z + \lambda\zeta = 0, \qquad z = 0, \tag{49}$$

$$[\zeta] = 0, \qquad [\zeta'_z] = 0, \qquad z = z_i, \tag{50}$$

$$\zeta'_z - \lambda\zeta = 0, \qquad z = z_N, \tag{51}$$

where the product $\mu_0\sigma$ is a piecewise constant function of z. The problem (48)–(51) can be characterized as a selfconjugate eigenvalue problem satisfying the Sturm oscillation theorem [27].

Thus, there are infinitely many eigenvalues, which are all real and can be arranged into a monotonically increasing unbounded sequence ($\alpha_0 < \alpha_1 < \dots < \alpha_j < \dots$). Each eigenvalue has multiplicity 1; thus, all eigenfunctions $\zeta_j(z)$, with the same eigenvalue α_j, differ from each other just by a constant nonzero factor. Each eigenfunction $\zeta_j(z)$ has exactly j zeroes in the open interval (z_1, z_N). The eigenfunctions satisfy the orthogonality condition

$$\int\limits_{z_1}^{z_N} \mu_0\sigma(z)\zeta_k(z)\zeta_j(z)\,dz = 0 \quad for \quad k \neq j. \tag{52}$$

The authors of [20] preferred a finite-difference solution to (51)–(58); whereas here we propose a continuation of the analytical solution. Thus, on the ith layer, we have a general solution of (48) in the form $\zeta(z) = A_i\exp(u_i z) + B_i\exp(-u_i z)$, where $u_i = \sqrt{\lambda^2 - \mu_0\sigma_i\alpha}$. This can be represented differently by expressing the coefficients A_i and B_i in terms of the values of the function and its derivatives at the ith boundary:

$$\zeta(z) = \zeta_i\,\mathrm{ch}\left[u_i(z - z_i)\right] + \frac{\zeta'_i}{u_i}\,\mathrm{sh}[u_i(z - z_i)]. \tag{53}$$

This representation provides us with a key for solving (48)–(51). Indeed, setting $\zeta_1 = 1$ at the upper boundary, we have $\zeta'_1 = -\lambda$ from (49). Now, using (53) as a recursive formula for calculating z_{i+1} from z_i, ζ_i, and ζ'_i, we obtain an equation for the eigenvalues α_j by means of the conditions on the lower boundary (51). In the process of solving this equation, all ζ_i and ζ'_i for $i = 2, \dots, N$ are determined; for each eigenvalue, by (53), the eigenfunction $\zeta_j(z)$ is determined completely in the domain $z_1 = 0 \geq z \geq z_N$. From now on, we use the index j to enumerate the eigenvalues and eigenfunctions; and the index i, to enumerate the layers in the cross-section. Therefore, ζ_{ji} is the value of the jth eigenfunction at the ith boundary. We also put $u_{ji} = \sqrt{\lambda^2 - \mu_0\sigma_i\alpha_j}$.

Now, we can represent the general solution of the quasistationary problem (47) as

$$X(z,t) = \sum_{j=0}^{\infty} C_j\zeta_j(z)\exp(-\alpha_j t), \tag{54}$$

where the coefficients C_j can be found from the initial condition

$$\overline{X}(z) = \sum_{j=0}^{\infty} C_j\zeta_j(z). \tag{55}$$

To determine C_j, we multiply (55) by $\mu_0\sigma\zeta_k(z)$ ($k = 0, 1, \dots$) and integrate with respect to z from zero to z_N. It can be verified by differentiation that we have the following antiderivatives:

$$\int \mu_0\sigma\zeta_j\zeta_k\,dz = \frac{1}{\alpha_k - \alpha_j}\left(\zeta'_j\zeta_k - \zeta_j\zeta'_k\right), \qquad \int \mu_0\sigma\overline{X}\zeta_k\,dz = \frac{1}{\alpha_k}\left(\overline{X}'_j\zeta_k - \overline{X}\zeta'_k\right),$$

$$\int \mu_0\sigma\zeta_j^2\,dz = \frac{\mu_0\sigma}{2u_j}\left\{z\left[u_j^2\zeta_i^2 + \left(\zeta'_j\right)^2\right] - \zeta'_j\zeta_j\right\}.$$

Inserting the limits for each homogeneous layer, summing, and using the conditions of the boundary value problems for $\overline{X}$ and ζ_j, we convince ourselves that the orthogonality relation holds also in the case of a piecewise constant function $\sigma(z)$; and we find for C_j that

$$C_j = \frac{\int\limits_0^{z_N} \mu_0 \sigma \overline{X} \zeta_j dz}{\int\limits_0^{z_N} \mu_0 \sigma \zeta_j^2 dz} = \frac{2\lambda \zeta_{jl}}{\mu_0 \alpha_j M_j}, \tag{56}$$

where

$$M_j = \sum_{i=1}^{N-1} \frac{\sigma_i}{2u_{ji}^2} \Big\{ d_i \big[\zeta_{ji}^2 u_{ji}^2 - \big(\zeta'_{ji}\big)^2\big] + \big(\zeta_{ji+1}\zeta'_{ji+1} - \zeta_{ji}\zeta'_{ji}\big) \Big\},$$

$d_i = z_i - z_{i+1}$ are the thickness of the layers; whereas, for ζ_{ji} and ζ'_{ji}, we have, in view of the general expression (53), the next recursive formulas:

$$\begin{aligned} \zeta_{ji+1} &= \zeta_{ji} \operatorname{ch}(u_{ji} d_i) - \frac{\zeta'_{ji}}{u_{ji}} \operatorname{sh}(u_{ji} d_i), \\ \zeta'_{ji+1} &= -\zeta_{ji} u_{ji} \operatorname{sh}(u_{ji} d_i) + \zeta'_{ji} \operatorname{ch}(u_{ji} d_i). \end{aligned} \tag{57}$$

It should be noticed that u_{ji} may assume imaginary values; and, in this event, a real hyperbolic solution transforms into a real trigonometric solution; i.e.,

$$\begin{aligned} \zeta_{ji+1} &= \zeta_{ji} \cos(u_{ji} d_i) - \frac{\zeta'_{ji}}{u_{ji}} \sin(u_{ji} d_i), \\ \zeta'_{ji+1} &= \zeta_{ji} u_{ji} \sin(u_{ji} d_i) + \zeta'_{ji} \cos(u_{ji} d_i), \end{aligned} \tag{58}$$

where $u_{ji} = \sqrt{\mu_0 \sigma_i \alpha_j - \lambda^2}$ now.

Thus, the problem is solved. Let us emphasize that the above algorithm has been quite successfully used already for more than ten years to calculate the stabilization curves in the PODBOR program complex for the electric exploration data processing by the sounding method of formation in the near zone (ZSB) in real time for the media with an arbitrary base, though the algorithm under consideration is the most convenient for the calculation of the late stages of the process in the media with an insulating base. The main difficulties of the numerical realization are connected with solving the transcendental equation for eigenvalues. However, this problem (of initial approximation) becomes much easier just because of the need to solve the equation for each node of integration in the integrals (44).

REFERENCES

1. V. I. Dmitriev, "A General Method for Calculating an Electromagnetic Field in a Stratified Medium," Vychisl. Metody i Programmirovanie, **10**, 55–65 (1968).
2. L. A. Tabarovskii, *Application of the Method of Integral Equations to the Problems of Geoelectricity* (Nauka, Novosibirsk, 1975) [in Russian].
3. L. L. Van'yan, *Fundamentals of Electromagnetic Sounding* (Nedra, Moscow, 1965) [in Russian].
4. J. R. Wait, *Geoelectromagnetism* (Academic, New York, 1982; Nedra, Moscow, 1987).
5. V. S. Mogilatov, "Inductive, Complex, and Galvanic Sources in Transient Electromagnetic Sounding," Izv. Ross. Acad. Nauk, Ser. Fizika Zemli, No. 12, 42–51 (1997).
6. V. S. Mogilatov, "Excitation of the Electromagnetic Field in a Stratified Earth by a Horizontal Current Sheet," Izv. Ross. Acad. Nauk, Ser. Fizika Zemli, No. 5, 45–53 (1998).
7. S. M. Sheinman, "On the Formation of Electromagnetic Fields in the Earth," Prikl. Geophiz. **9** (Gostoptekhizdat, Moscow, 1947), pp. 3–55.
8. A. N. Tikhonov and O. A. Skugarevskaya, "On the Electric Current Formation in an Unhomogeneous Medium. II," Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz. **14** (4), 281–293 (1950).
9. A. N. Tikhonov, "On the Electric Current Formation in a Homogeneous Conductive Half-Space," Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz. **10** (3), 213–231 (1946).

10. L. A. Tabarovskii and V. P. Sokolov, "A Program for Calculation of the Nonstationary Field from Dipole Sources in a Horizontal Stratified Medium (ALEKS)," in *Electromagnetic Methods of Geophysical Studies* (Inst. Geolog. i Geofiz., Novosibirsk, 1982), pp. 57–77.
11. W. L. Anderson, "Improved Digital Filters for Evaluating the Fourier and Hankel Transform Integrals," Report PB-242-156 (U. S. Dept. of Commerce, Nat. Tech. Inform. Service, 1975).
12. W. L. Anderson, "Computer Program: Numerical Integration of Related Hankel Transforms of Orders 0 and 1 by Adaptive Digital Filtering," Geophysics **44** (7), 1287–1305 (1979).
13. W. L. Anderson, "Computation of Green's Tensor Integrals for Three-Dimensional Electromagnetic Problems Using Fast Hankel Transforms," Geophysics **49** (10), 1754–1759 (1984).
14. W. L. Anderson, "A Hybrid Fast Hankel Transform Algorithm for Electromagnetic Modeling," Geophysics **54** (2), 263–266 (1989).
15. O. A. Skugarevskaya, "Calculation of the Final Stage of the Process of Electric Field Formation in a Three-Layer Medium," Izv. Akad. Nauk SSSR, Ser. Geofiz., No. 1, 59–72 (1959).
16. V. I. Dmitriev, O. A. Skugarevskaya, and P. P. Frolov, *Some Problems of the Method of Field Formation in a Near Zone* (Moskov. Gos. Univ., Moscow, 1973) [in Russian].
17. E. Prácser, "Fast Computing of Transient Electromagnetic Field on the Surface of a Layered Half-Space," Geofizik. Kozlemenyek **37** (2–3), 159–176 (1992).
18. A. N. Tikhonov, "On Electric Current Formation in an Unhomogeneous Stratified Medium," Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz. **14** (3), 199–222 (1950).
19. M. M. Gol'dman, "The Integral Finite Difference Method for Calculating the Transient Electromagnetic Fields in a Horizontally-Stratified Medium," Geophys. Prospecting **31** (4), 664–686 (1983).
20. M. I. Epov and I. N. Yel'tsov, "Relaxation of the Electromagnetic Field from a Dipole Source in a Conductive Stratified Layer Submerged in Insulator," Geologiya i Geofizika, No. 10, 126–129 (1991).
21. L. A. Tabarovskii and M. I. Epov, "Discrete Spectra in the Problems of Nonstationary Field Difraction on the Sheinman Films," Izv. Ros. Akad. Nauk, Ser. Fizika Zemli, No. 9, 46–54 (1998).
22. M. I. Epov and I. N. Yel'tsov, *Direct and Inverse Problems of Inductive Geoelectricity in One-Dimensional Media* (Trofimuk United Inst. of Geology, Geophysics, and Mineralogy, Novosibirsk, 1992) [in Russian].
23. V. S. Mogilatov, "Late Stage of Formation of the Electromagnetic Field Excited by a Submerged Electric Dipole," Izv. Akad. Nauk SSSR, Ser. Fizika Zemli, No. 8, 103–107 (1976).
24. M. M. Gol'dman and V. S. Mogilatov, "Formation of the Field of a Vertical Electric Dipole Submerged in a Horizontally Stratified Half-Space," in *Theory and Practice of the Electromagnetic Field Application in the Exploration Geophysics* (Inst. Geolog. i Geofiz., Novosibirsk, 1978), pp. 123–138.
25. B. I. Rabinovich and V. S. Mogilatov, "Formation of the Field of a Submerged Vertical Magnetic Dipole," Geologiya i Geofizika, No. 3, 88–100 (1981).
26. V. S. Mogilatov, "On a Method for Solving the Basic Forward Problem of the Transient Field Electric Exploration," Geologiya i Geofizika, No. 3, 108–117 (1993).
27. E. Kamke, *Handbook of Ordinary Differential Equations* (Leipzig, 1959; Nauka, Moscow, 1976).